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On a Minimal Property of Cubic Periodic Lagrangian Splines

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We prove the following theorem concerning cubic periodic spline interpolation: If l_j denote the Lagrangian splines in cubic periodic spline interpolation with period N on the grid $\mathbb{Z}$, then the sum of the squares of the l_j, $j = 0,..., N-1$, is bounded by one. An analogous result for the space $\mathbb{P}_n$ of algebraic polynomials of degree n and for the interval $[-1, 1]$ was given by Fejér.

Let $S_m^{(N)}$ be the space of periodic splines of degree m with period N on the grid $\mathbb{Z}$ ($N, m \in \mathbb{N}$, m odd). Given N values $y_0,..., y_{N-1} \in \mathbb{R}$, there exists a unique spline $s \in S_m^{(N)}$, for which $s(j) = y_j$, $j = 0,..., N-1$.

Let $l_j^{(N)} \in S_m^{(N)}$ be the jth Lagrange spline, i.e., $l_j^{(N)}(i) = \delta_{ij}$, $i, j \in \{0,..., N-1\}$. By a result of Reimer [5],

$$\|l_j^{(N)}\| = \max_{x \in \mathbb{R}} |l_j^{(N)}(x)| = 1; \tag{1}$$

this means that $\{l_j^{(N)}: j = 0,..., N-1\}$ form an extremal basis of $S_m^{(N)}$ (for the definition of an extremal basis see Reimer [4]).

An analogous result for the space $\mathbb{P}_n$ of algebraic polynomials of degree n was given by Fejér [1]. The nodes used by Fejér are the relative extremals of the Legendre polynomial of degree n. He even proved the following inequality for the corresponding Lagrange polynomials $\tilde{l}_j$, $j = 0,..., n$:

$$\sum_{j=0}^{n} (\tilde{l}_j(x))^2 \leqslant 1, \qquad x \in [-1, 1]. \tag{2}$$

By (1) and the boundedness of the Lebesgue constants $\|\sum_{j=0}^{N-1} |l_j^{(N)}|\|$ (see Richards [6]), the norm of $\sum_{j=0}^{N-1} (l_j^{(N)})^2$ is bounded too. Because of this and Fejér's result, we even conjecture that

$$s^{(N)}(x) := \sum_{j=0}^{N-1} (l_j^{(N)}(x))^2 \leqslant 1, \qquad x \in \mathbb{R}. \tag{3}$$

In what follows we prove this conjecture for $m = 3$.

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THEOREM. *Let $l_j^{(N)} \in S_3^{(N)}$, $j = 0,\dots, N-1$, be the cubic periodic Lagrange splines corresponding to the nodes $0,\dots, N-1$ (i.e., $l_j^{(N)}(i) = \delta_{ij}$). Then*

$$s^{(N)}(x) = \sum_{j=0}^{N-1} (l_j^{(N)}(x))^2 \leqslant 1, \qquad x \in \mathbb{R}. \tag{4}$$

Before proving the Theorem, we derive an explicit expression for $s^{(N)}$. Let $q_j(t) := l_0^{(N)}(j+t)$, $t \in \mathbb{R}$, $j \in \mathbb{Z}$, be the "shifted" Lagrange spline $l_0^{(N)}$ and let $H_m(t, z)$ denote the generalized Euler–Frobenius polynomial of degree m (for a definition see Meinardus and Merz [2], ter Morsche [3], and Reimer [5]). Further, let

$$A_\mu(t) := H_m(t, z_\mu), \quad B_\mu(t) := A_\mu(1-t), \quad C_\mu := H_m^z(0, z_\mu), \quad \mu = 1,\dots, r,$$

where $m = 2r+1$, z_μ are those zeros of $H_m(0, \cdot)$ with $z_1 < \cdots < z_r < -1$ and H_m^z denotes the partial derivative of H_m with respect to the second variable. By a result of Reimer [5] we have

$$q_j^{(N)}(t) = (1-t)^m \delta_{0,j} + t^m \delta_{N-1,j} - \sum_{\mu=1}^{r} \frac{H_m(t, z_\mu)\, z_\mu^{N-j-1} + H_m(1-t, z_\mu)\, z_\mu^j}{H_m^z(0, z_\mu)(z_\mu^N - 1)} \tag{5}$$

for $t \in [0, 1]$, $j \in \{0, 1,\dots, N-1\}$. Hence for $N > 1$

$$\begin{aligned} s^{(N)}(t) = {} & \sum_{\mu=1}^{r} \frac{1}{(z_\mu - 1)^2 C_\mu^2} \cdot \frac{z_\mu^N + 1}{z_\mu^N - 1} (A_\mu^2(t) + B_\mu^2(t)) \\ & + 2N \cdot \sum_{\mu=1}^{r} \frac{z_\mu^{N-1}}{C_\mu^2 (z_\mu^N - 1)^2} \cdot A_\mu(t) \cdot B_\mu(t) \\ & + 2 \cdot \sum_{1 \leqslant \mu < \nu \leqslant r} \frac{1}{C_\mu C_\nu (z_\mu^N - 1)(z_\nu^N - 1)} \\ & \times \left\{ \alpha_{\mu\nu}(t) \frac{z_\mu^N z_\nu^N - 1}{z_\mu z_\nu - 1} + \beta_{\mu\nu}(t) \frac{z_\mu^N - z_\nu^N}{z_\mu - z_\nu} \right\} \\ & + (1-t)^{2m} + t^{2m} - 2 \cdot (1-t)^m \cdot \sum_{\mu=1}^{r} \frac{A_\mu(t)\, z_\mu^{N-1} + B_\mu(t)}{C_\mu (z_\mu^N - 1)} \\ & - 2t^m \cdot \sum_{\mu=1}^{r} \frac{A_\mu(t) + B_\mu(t)\, z_\mu^{N-1}}{C_\mu (z_\mu^N - 1)}, \end{aligned} \tag{6}$$

where

$$\alpha_{\mu\nu}(t) := A_\mu(t) \cdot A_\nu(t) + B_\mu(t) \cdot B_\nu(t)$$

and

$$\beta_{\mu\nu}(t) := A_\mu(t) \cdot B_\nu(t) + A_\nu(t) \cdot B_\mu(t). \tag{7}$$

Proof of the Theorem. Let $p^{(N)}$ be the unique polynomial of degree 6 with $p^{(N)}|_{[0,1]} = s^{(N)}|_{[0,1]}$ and let t_0 be the greatest zero of $H_3(\cdot, z_1)$. By a short calculation, $t_0 = (1+\sqrt{3})/2$. Writing A, B, C instead of A_1, B_1, C_1 and using (6), we have for $N > 1$

$$\begin{aligned} p^{(N)}(t) = {} & \frac{A^2(t)+B^2(t)}{C^2(z_1^2-1)} \cdot \left(1 + \frac{2}{z_1^N - 1}\right) + N \frac{2 \cdot A(t) \cdot B(t)}{C^2} \cdot \frac{z_1^{N-1}}{(z_1^N-1)^2} \\ & + (1-t)^6 + t^6 - \frac{2}{C(z_1^N-1)} \\ & \times ((1-t)^3(A(t)\, z_1^{N-1} + B(t)) + t^3(A(t) + B(t)\, z_1^{N-1})). \end{aligned} \tag{8}$$

In particular, for $t = t_0$, (8) yields for $N > 1$

$$p^{(N)}(t_0) = \frac{2B(t_0)}{C(z_1^N-1)} \cdot \frac{\sqrt{3}-1}{4} + 2 - \frac{3}{4}\sqrt{3}$$

and (9)

$$p^{(\infty)}(t_0) := \lim_{N\to\infty} p^{(N)}(t_0) = 2 - \frac{3}{4}\sqrt{3} < 1.$$

Since $B(t_0) < 0$ and $C > 0$ formula (9) implies the inequalities

$$p^{(2)}(t_0) < p^{(4)}(t_0) < \cdots < p^{(2N)}(t_0) < p^{(\infty)}(t_0)$$

and (10)

$$p^{(1)}(t_0) > p^{(3)}(t_0) > \cdots > p^{(2N+1)}(t_0) > p^{(\infty)}(t_0).$$

Because of $p^{(1)} \equiv 1$, formulas (9) and (10) together imply

$$p^{(N)}(t_0) \leqslant 1 \qquad \text{for all } N \in \mathbb{N}. \tag{11}$$

As a consequence of (8) we have $p^{(N)}(t) = p^{(N)}(1-t)$, $t \in \mathbb{R}$. This, together with $p^{(N)}(0) = 1$, $(d/dt)\, p^{(N)}(0) = 0$ (note that $s^{(N)}(t) = s^{(N)}(-t)$) and (11), implies that $(d/dt)\, p^{(N)}$ has zeros at $1-\xi$, 0, $\frac{1}{2}$, 1, ξ, where $\xi > 1$. Since $(d/dt)\, p^{(N)}$ has degree 5, these are the only zeros and they are simple. Using $\lim_{t\to\pm\infty} p^{(N)}(t) = +\infty$ we conclude that $p^{(N)}$ has relative maxima at $t = 0$ and $t = 1$ and hence a relative minimum at $t = \frac{1}{2}$. Thus

$$p^{(N)}(t) \leqslant 1, \qquad t \in [0, 1]. \tag{12}$$

As $l_j^{(N)}(t) = l_{j+1}^{(N)}(t+1)$, $s^{(N)}$ has period 1. Since $p^{(N)}|_{[0,1]} = s^{(N)}|_{[0,1]}$, (12) completes the proof.

As the Lagrange splines $l_j^{(N)}$ converge to the cardinal Lagrange spline $l_j^{(\infty)}$, we have the following

Corollary. *Let $l_j^{(\infty)}$ be the jth cardinal Lagrange spline for the grid $\mathbb{Z}$ (i.e., $l_j^{(\infty)} = \delta_{ij}$, $i, j \in \mathbb{Z}$) of degree 3. Then*

$$\left\| \sum_{j \in \mathbb{Z}} (l_j^{(\infty)})^2 \right\| = 1.$$

Proof. Let $N_0, N_1 \in \mathbb{N}$ be arbitrary. Then by the Theorem,

$$\sum_{j=0}^{N_0+N_1} (l_j^{(N)}(t))^2 = \sum_{j=-N_0}^{N_1} (l_{j+N_0}^{(N)}(t))^2 = \sum_{j=-N_0}^{N_1} (l_j^{(N)}(t-N_0))^2 \leqslant 1$$

for $N > N_0 + N_1$, $t \in \mathbb{R}$. Hence

$$\sum_{j=-N_0}^{N_1} (l_j^{(\infty)}(t))^2 = \lim_{N \to \infty} \sum_{j=-N_0}^{N_1} (l_j^{(N)}(t))^2 \leqslant 1,$$

which proves the Corollary.

As a consequence of the Theorem and the Corollary we have

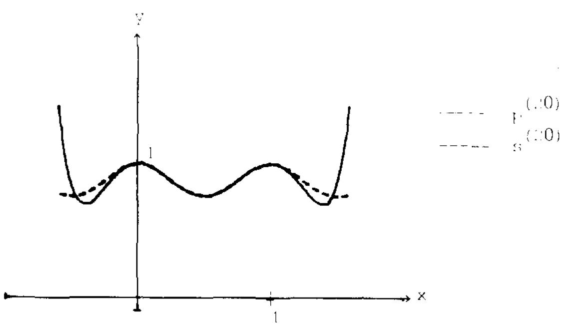
$$\| l_j^{(N)} \| = 1, \qquad j = 0, \ldots, N-1, \tag{13}$$

and

$$\| l_j^{(\infty)} \| = 1, \qquad j \in \mathbb{Z}. \tag{14}$$

This result is part of a theorem of Reimer [5], who proved (13) and (14) for any periodic Lagrangian spline of odd degree m.

The following graph illustrates the quantitative behaviour of $s^{(N)}$ and $p^{(N)}$ for the typical case $N = 20$.



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